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ON CENTRES AND LINES OF MEAN POSITION.

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§ 1. INTRODUCTORY.

In this paper we shall define the centre of mean position of a number of points in a plane, with respect to a line of the plane, to be the centre of gravity of the points, supposing them to have weights inversely proportional to their distances from the line, points on one side of the line having positive and those on the other negative weights. Let us take a triangle of reference of which the line in question shall be one of the sides, and calling it the axis of z, let the axes of x and y be any two other straight lines. Calling the co-ordinates of the n points $x_1, y_1, z_1, \ldots x_n, y_n, z_n$, and those of the centre of mean position X, Y, Z, we have

$$X = \frac{\frac{x_1}{z_1} + \ldots + \frac{x_n}{z_n}}{\frac{1}{z_1} + \ldots + \frac{1}{z_n}},$$

$$Y = \frac{\frac{y_1}{z_1} + \ldots + \frac{y_n}{z_n}}{\frac{1}{z_1} + \ldots + \frac{1}{z_n}},$$

$$Z = \frac{\frac{z_1}{z_1} + \ldots + \frac{z_n}{z_n}}{\frac{1}{z_1} + \ldots + \frac{1}{z_n}};$$

or

$$\frac{X}{Z} = \frac{\frac{x_1}{z_1} + \ldots + \frac{x_n}{z_n}}{n},$$

$$\frac{Y}{Z} = \frac{\frac{y_1}{z_1} + \ldots + \frac{y_n}{z_n}}{n};$$

which latter expression we shall use. If the straight line of reference be the line at infinity, the centre of mean position is the centre of gravity of the points, supposing them all to have equal weights.

The same thing can be done for lines. We will take three points of reference $\alpha = 0$, $\beta = 0$, and $\gamma = 0$ in tangential co-ordinates, and will take the co-ordinates of any line to be the perpendiculars from the three points upon the line; taking the perpendicular from $\gamma = 0$ positive, and the perpendiculars from the others positive or negative according as they lie on the same side of the line with $\gamma = 0$ or on the opposite. We will define the co-ordinates of the mean line of position with respect to $\gamma = 0$ of any lines $\alpha_1, \beta_1, \gamma_1, \ldots \alpha_n, \beta_n, \gamma_n$, to be

$$\frac{\alpha}{\Gamma} = \frac{\sum_1^n \frac{\alpha}{\gamma}}{n}, \qquad \frac{\beta}{\Gamma} = \frac{\sum_1^n \frac{\beta}{\gamma}}{n}.$$

§ 2.

This being premised, let

$$x^n f(u) + x^{n-1} z f_1(u) + x^{n-2} z^2 f_2(u) + \ldots = 0$$

be the equation of a plane curve of the nth degree in trilinear co-ordinates, having written $u = y/x$. If we wish to get the tangents to the curve where the axis of z crosses it, we proceed as follows: Substitute

$$\frac{y}{x} = a + a' \frac{z}{x},$$

and determine a and a' so that the resulting equation in $\frac{x}{z}$ may have two infinite roots. The equation in x/z will be

$$\left(\frac{x}{z}\right)^n f(a) + \left(\frac{x}{z}\right)^{n-1} \Big[a' f'(a) + f_1(a)\Big]$$
$$+ \left(\frac{x}{z}\right)^{n-2} \left[\frac{a'^2}{2} f''(a) + a' f'_1(a) + f_2(a)\right] + \ldots = 0,$$

where we have written $f'(a) = df(a)/da$. In order that the equation in x/z may have two infinite roots, we must have

$$f(a) = 0 \quad \text{and} \quad a' f'(a) + f_1(a) = 0, \tag{1}$$

which will determine in general both a and a'. If it happen that the axis of z cut the curve in n points of inflexion, we shall also have

$$a'^2 f''(a) + 2a' f_1'(a) + 2 f_2(a) = 0$$

for all the sets of values of a and a'.

Let us now consider the points of contact of tangents from any point x', y', z' in the plane to the curve. They will be the intersections of the curve $U = 0$ with the first polar of the point; namely,

$$x'\frac{\partial U}{\partial x} + y'\frac{\partial U}{\partial y} + z'\frac{\partial U}{\partial z} = 0\,;$$

or we may write it

$$\left[\frac{x}{z}\right]^{n-1} F(u) + \left[\frac{x}{z}\right]^{n-2} F_1(u) + \ldots = 0,$$

where

$$F(u) = nx'f(u) + (y' - ux')f'(u) + z'f_1(u)$$

and

$$F_1(u) = (n-1)\,x'f_1(u) + (y' - ux')\,f_1'(u) + 2z'f_2(u),$$

where $u = y/x$.

Now Liouville has shown that if we eliminate y between this equation and the equation of the curve we shall get

$$-\Sigma\frac{x}{z} = \Sigma\frac{a'F'(a) + F_1(a)}{F(a)}, \tag{2}$$

where the second Σ pertains to all the sets of values of a and a' derived from the equations

$$f(a) = 0 \quad \text{and} \quad a'f'(a) + f_1(a) = 0.$$

Substituting for F and F_1 their value in (2), we have

$$-\Sigma\frac{x}{z} = \Sigma\frac{(y' - ax')\,(a'f''(a) + f_1'(a)) + z'\,(a'f_1'(a) + 2f_2(a))}{f'(a)\,(y' - ax') + z'f_1(a)}$$

$$= \Sigma\frac{a'f''(a) + f_1'(a)}{f'(a)} + \Sigma\frac{a'^2f''(a) + 2a'f_1'(a) + 2f_2(a)}{f'(a)(y' - ax') + z'f_1(a)}\,z'.$$

If the curve have n points of inflexion on the axis of z, we shall have

$$a'^2f''(a) + 2a'f_1'(a) + 2f_2(a) = 0$$

for all the sets of values of a and a', for they are determined by the same equations (1) as before. In this case we see that $\Sigma\,(x/z)$ is constant and independent of the point prime from which the tangents are drawn. We shall take

the axis of z to be the line to which the centre of mean position of the points of contact is referred, and therefore we shall have

$$\frac{X}{Z} = \frac{\Sigma \frac{x}{z}}{n(n-1)} \text{ and } \frac{Y}{Z} = \frac{\Sigma \frac{y}{z}}{n(n-1)}.$$

We have therefore the theorem, that if a curve of the nth degree have n points of inflexion upon a right line, the mean centre of position with respect to that line, of the points of contact of tangents from any point in the plane to the curve, will be independent of the point.

By using line co-ordinates we would have obtained the theorem: If the tangents at n cuspidal points on a curve of the nth class meet in a point, the mean line of position with respect to this point of the tangents at the points where any line crosses the curve, will be independent of the line.

Every non-singular and nodal cubic has three points of inflexion upon a right line, and therefore we have as a corollary the theorem, that if tangents be drawn from any point in the plane to a non-singular or nodal cubic, the mean centre of position of the points of contact with reference to the line upon which three inflexions lie, is independent of the point. It may be interesting perhaps to see what this point is for the cubics in question. Every non-singular cubic can be put in the form

$$x^3 + y^3 + z^3 + 6mxyz = 0,$$

each of the axes of reference cutting the cubic in three points of inflexion. In this case we have

$$f(a) = a^3 + 1, \quad f_1(a) = 6ma, \quad f_2(a) = 0.$$

a' is determined by the equation

$$3a^2a' + 6ma = 0,$$

or

$$a' = -\frac{2m}{a}.$$

We have then

$$n(n-1)\frac{X}{Z} = -\Sigma \frac{a'f''(a) + f_1'(a)}{f'(a)}$$

$$= -\Sigma \frac{-\frac{12m}{a}a + 6m}{3a^2}$$

$$= 2m\Sigma \frac{1}{a^2} = 2m\Sigma a^2 = 0.$$

Likewise

$$\frac{Y}{Z} = 0.$$

We have therefore the theorem, that the common mean centre of position of the points of contact of tangents from any point in the plane to a non-singular cubic, with reference to a line upon which three inflexions lie, coincides with the point of intersection of the two lines upon which the other six inflexions lie. Also, that the line joining any point on this cubic with the centre of mean position of the points of contact of the four tangents (other than the tangent at the point itself) from it to the curve will pass through a fixed point. In the case of the nodal cubic the common mean centre is readily seen to be the node itself. For if we consider the points of contact of tangents from it to the curve, we see that all six points coincide with the node itself. From any point on a nodal cubic can be drawn two tangents to the curve other than the tangent at the point. Therefore the two tangents from a point on a nodal cubic to the curve, and the two lines joining the point with the node and with the point where the line joining the points of contact cuts the line upon which the three inflexions lie, form an harmonic pencil.

From the above follows the reciprocal theorem. Every curve of the 3rd class which possesses neither double tangents nor inflexions, has 9 cusps the tangents at which meet by threes in twelve points. If we take the point where any three meet as the point of reference, then the mean line of position with respect to it of the tangents at the points where any line crosses the curve coincides with the line joining the pair of points in which the other six cuspidal tangents meet.

§ 3.

We have found the following expression for the mean centre of position of the points of contact of tangents from the point x', y', z', to the curve with respect to the axis of z:

$$-n(n-1)\frac{X}{Z} = \Sigma\frac{a'_{\cdot}f''(a) + f_1'(a)}{f'(a)} + \Sigma\frac{a'^2_{\cdot}f'''(a) + 2a'_{\cdot}f_1''(a) + 2f_2'(a)}{f'(a)(y' - ax') + z'f_1'(a)}z'.$$

This expression will be constant for all points on the axis of z, since z' is a factor in the variable part. Therefore we have the theorem, that if tangents be drawn from any point on a given line to a curve of any degree, the mean centre of position of the points of contact with reference to that line will be independent of the point. This is an extension of the theorem of Chasles that "the centre of gravity of the points of contact of parallel tangents to any curve is independent of the direction of the tangents," since when the given line is at infinity the mean centre of position with regard to it coincides with the centre of gravity of the points supposing them to have equal weights, and the tangents from any point on it are parallel.

We will now find the mean centre of position of the poles of a line with respect to the line. We will choose the line for the axis of z. Let

$$U = x^n f(u) + x^{n-1} z f_1(u) + \ldots = 0$$

be the equation of a curve of the nth degree, then will the poles of the axis of z with respect to the curve be given by the solution of the equations

$$\frac{\partial U}{\partial x} - \frac{u}{x}\frac{\partial U}{\partial u} = 0 \qquad \text{and} \qquad \frac{1}{x}\frac{\partial U}{\partial u} = 0,$$

which are the first polars of the points where the axes of x and y cross the axis of z.

These equations are equivalent to

$$x^{n-1}(nf - uf') + x^{n-2}z\left((n-1)f_1 - uf_1'\right) + \ldots = 0$$

and

$$x^{n-1}f' + x^{n-2}zf_1' + \ldots = 0.$$

Writing $\qquad F = nf - uf', \qquad F_1 = (n-1)f_1 - uf_1;$

we have

$$-\Sigma \frac{x}{z} = \Sigma \frac{a'F'(a) + F_1(a)}{F(a)},$$

where a and a' are determined by the equations

$$f'(a) = 0 \quad \text{and} \quad a'f''(a) + f_1'(a) = 0.$$

Substituting the values of F and F_1 we have

$$-\Sigma \frac{x}{z} = \Sigma \frac{a'\left[(n-1)f' - af''\right] + (n-1)f_1 - af_1'}{nf - af'}$$

$$= \frac{n-1}{n}\Sigma \frac{f_1(a)}{f(a)};$$

or, since $\qquad \dfrac{X}{Z} = \dfrac{\Sigma \frac{x}{z}}{(n-1)^2}, \qquad -\dfrac{X}{Z} = \dfrac{\Sigma \frac{f_1(a)}{f(a)}}{n(n-1)}.$

The mean centre of position of the points of contact of tangents from any point in the line with respect to the line might have been obtained in a different form by reversing the process of elimination. Thus the two equations being

$$x^n f + x^{n-1}zf_1 + \ldots = 0$$

and

$$x^{n-1}f' + x^{n-2}zf_1' + \ldots = 0,$$

we have

$$-\Sigma\frac{x}{z}=\Sigma\frac{a'f'''(a)+f_1(a)}{f'(a)},$$

where a and a' are determined by the equations

$$f'(a)=0 \quad \text{and} \quad a'f''(a)+f_1(a)=0,$$

or

$$-\Sigma\frac{x}{z}=\Sigma\frac{f_1(a)}{f(a)};$$

$$\therefore\ -\frac{X}{Z}=\frac{1}{n(n-1)}\Sigma\frac{f_1'(a)}{f'(a)};$$

and since a is determined by the same equation as before, we have the theorem that the mean centre of position of the points of contact of tangents to a curve from any point on a given line with respect to that line coincides with the mean centre of position with respect to the line of the poles of the line with respect to the curve. This point can readily be shown to be the tangential pole of the line.

In tangential co-ordinates any point has $(n-1)^2$ polar lines. We have as a reciprocal theorem then, that the mean line of position with respect to a point of the polar lines of the point with respect to the curve coincides with the mean line of position with respect to the point, of the tangents at the points where any line through the given point cuts the curve. This mean polar, reciprocally, is readily seen to be the trilinear polar line of the point.

§ 4.

These ideas may be readily extended to space. Let us define the mean centre of position of any number of points in space with respect to a plane, to be the centre of gravity of the points supposing them to have weights inversely proportional to their distances from the plane, points on one side having positive, and those on the other, negative weights. As before, calling the co-ordinates of the points $x_1, y_1, z_1, w_1, \ldots x_n, y_n, z_n, w_n$, and those of the centre of mean position X, Y, Z, W, we shall have

$$\frac{X}{W}=\frac{\frac{x_1}{w_1}+\ldots+\frac{x_n}{w_n}}{n},$$

$$\frac{Y}{W}=\frac{\frac{y_1}{w_1}+\ldots+\frac{y_n}{w_n}}{n},$$

$$\frac{Z}{W}=\frac{\frac{z_1}{w_1}+\ldots+\frac{z_n}{w_n}}{n}.$$

Let $U = \left[\frac{x}{w}\right]^n f(u, v) + \left[\frac{x}{w}\right]^{n-1} f_1(u, v) + \left[\frac{x}{w}\right]^{n-2} f_2(u, v) + \ldots = 0,$

where
$$u = \frac{y}{x} \quad \text{and} \quad v = \frac{z}{x},$$

be the equation of a surface of the nth degree in homogeneous co-ordinates. The first polar of the intersection of the x, y, and z planes with respect to the surface is

$$\left[\frac{x}{w}\right]^{n-1} f_1(u, v) + 2\left[\frac{x}{w}\right]^{n-2} f_2(u, v) + \ldots = 0.$$

We get the tangents to the curve of intersection at the points where it crosses the w plane as follows:

Let

$$\frac{y}{x} = \alpha + \alpha' \frac{w}{x}, \qquad \frac{z}{x} = \beta + \beta' \frac{w}{x}$$

be the equations of any line. Substituting these values in the equation of the surfaces, we shall have

$$x^n f(\alpha, \beta) + wx^{n-1}\left[\alpha' \frac{\partial f}{\partial \alpha} + \beta' \frac{\partial f}{\partial \beta} + f_1\right] + w^2 x^{n-2}\left[\frac{\alpha' A + \beta' B + C}{2}\right] + \ldots = 0,$$

where

$$A = \alpha' \frac{\partial^2 f}{\partial \alpha^2} + \beta' \frac{\partial^2 f}{\partial \alpha \partial \beta} + \frac{\partial f_1}{\partial \alpha},$$

$$B = \alpha' \frac{\partial^2 f}{\partial \alpha \partial \beta} + \beta' \frac{\partial^2 f}{\partial \beta^2} + \frac{\partial f_1}{\partial \beta},$$

$$C = \alpha' \frac{\partial f_1}{\partial \alpha} + \beta' \frac{\partial f_1}{\partial \beta} + 2 f_2,$$

and

$$x^{n-1} f_1(\alpha, \beta) + x^{n-2} w\left[\alpha' \frac{\partial f_1}{\partial \alpha} + \beta' \frac{\partial f_1}{\partial \beta} + 2 f_2\right] + \ldots = 0;$$

understanding any f written without anything following it to be expressed in terms of α and β.

In order that the line may touch the curve of intersection at the w plane, α, β, α', and β' are determined by the equations

$$f(\alpha, \beta) = 0, \qquad f_1(\alpha, \beta) = 0,$$

$$\alpha' \frac{\partial f}{\partial \alpha} + \beta' \frac{\partial f}{\partial \beta} + f_1 = 0, \quad \text{and} \quad \alpha' \frac{\partial f_1}{\partial \alpha} + \beta' \frac{\partial f_1}{\partial \beta} + 2 f_2 = 0.$$

But if the line touches the surface $U = 0$ in three points at every one of the points where the curve crosses the w plane, we shall also have

$$\alpha' A + \beta' B = 0$$

for all the sets of values of α, α', β, and β'.

This being premised, let x', y', z', w' be any point in space. The points of contact of tangent planes through the line joining x', y', z', w' with the intersection of the x, y, and z planes, will be given by the solution of the equations

$$U = x^n f(u, v) + w x^{n-1} f_1(u, v) + w^2 x^{n-2} f_2(u, v) + \ldots = 0,$$

$$x^{n-1} f_1(u, v) + 2 w x^{n-2} f_2(u, v) + \ldots = 0,$$

and

$$x^{n-1} \varphi(u, v) + w x^{n-2} \varphi_1(u, v) + \ldots = 0;$$

where

$$\varphi(u, v) = n x' f + (y' - u x') \frac{\partial f}{\partial u} + (z' - v x') \frac{\partial f}{\partial v} + w' f_1,$$

$$\varphi_1(u, v) = (n-1) x' f_1 + (y' - u x') \frac{\partial f_1}{\partial u} + (z' - v x') \frac{\partial f_1}{\partial v} + 2 w' f_2.$$

From these equations we have

$$-\Sigma \frac{x}{w} = \Sigma \frac{\alpha' \dfrac{\partial \varphi}{\partial \alpha} + \beta' \dfrac{\partial \varphi}{\partial \beta} + \varphi_1}{\varphi},$$

where the second Σ extends to all the sets of values of α, α', β, and β' derived from the equations

$$f(\alpha, \beta) = 0, \qquad f_1(\alpha, \beta) = 0,$$

$$\alpha' \frac{\partial f}{\partial \alpha} + \beta' \frac{\partial f}{\partial \beta} = 0, \quad \text{and} \quad \alpha' \frac{\partial f_1}{\partial \alpha} + \beta' \frac{\partial f_1}{\partial \beta} + 2 f_2 = 0.$$

Substituting the values of φ and φ_1 we have

$$-\Sigma \frac{x}{w} = \Sigma \frac{(y' - \alpha x') A + (z' - \beta x') B}{(y' - \alpha x') \dfrac{\partial f}{\partial \alpha} + (z' - \beta x') \dfrac{\partial f}{\partial \beta}}$$

$$= \Sigma \frac{A}{\dfrac{\partial f}{\partial \alpha}} + \Sigma \frac{(\alpha' A + \beta' B)(z' - \beta x')}{\left[(y' - \alpha x') \dfrac{\partial f}{\partial \alpha} + (z' - \beta x') \dfrac{\partial f}{\partial \beta}\right] \beta'},$$

where A and B have the same values as given above. If the condition mentioned above is fulfilled, we shall have $\alpha' A + \beta' B = 0$ for all the sets of values of α, α', β, and β', since they are determined by exactly the same equations as

before. Therefore, if we define the co-ordinates of the mean centre by the equations

$$n\,(n-1)^2\,\frac{X}{W}=\Sigma\,\frac{x}{w},$$

and similarly for Y and Z, we shall have the following theorem:

If the first polar of a point with respect to a surface cut the surface at $n\,(n-1)$ points in a plane along the asymptotic lines, the mean centre of position with reference to this plane of the points of contact of tangent planes to the surface through any line through this point, will be independent of the line.

§ 5.

The extension of the theorem concerning parallel tangent planes to a surface may be obtained as follows: Let

$$x^n f(u,v)+x^{n-1}wf_1(u,v)+x^{n-2}w^2f_2(u,v)+\ldots=0$$

be the equation of a surface of the nth degree. The first polar of the intersection of the x, z, and w planes will be

$$x^{n-1}\frac{\partial f}{\partial u}+x^{n-2}w\frac{\partial f_1}{\partial u}+\ldots=0.$$

Let us take any other point in the w plane x', y', z', 0. The co-ordinates of the points of contact of tangent planes through the line joining these two points will be given by the solution of the above equations and the first polar of x', y', z', 0 with respect to the surface; namely,

$$x^{n-1}\varphi\,(u,v)+x^{n-2}w\varphi_1\,(u,v)+\ldots=0,$$

where

$$\varphi=nx'f+(y'-ux')\frac{\partial f}{\partial u}+(z'-vx')\frac{\partial f}{\partial v},$$

and

$$\varphi_1=(n-1)\,x'f_1+(y'-ux')\frac{\partial f_1}{\partial u}+(z'-vx')\frac{\partial f_1}{\partial v}.$$

From these we get

$$-\Sigma\,\frac{x}{w}=\Sigma\,\frac{\alpha'\dfrac{\partial\varphi}{\partial\alpha}+\beta'\dfrac{\partial\varphi}{\partial\beta}+\varphi_1}{\varphi},$$

where α, β, α', and β' are given by the equations

$$f(\alpha,\beta)=0,\qquad \alpha'\frac{\partial f}{\partial\alpha}+\beta'\frac{\partial f}{\partial\beta}+f_1=0,$$

$$\frac{\partial f}{\partial\alpha}=0,\quad\text{and}\quad \alpha'\frac{\partial^2 f}{\partial\alpha^2}+\beta'\frac{\partial^2 f}{\partial\alpha\partial\beta}+\frac{\partial f_1}{\partial\alpha}=0.$$

Substituting the values of φ and φ_1, we shall find the multipliers of x' and $y' - \alpha x'$ in both numerator and denominator of $\Sigma(x/w)$ vanish, and we have

$$-\Sigma\frac{x}{w} = \Sigma\frac{\left[\alpha'\frac{\partial^2 f}{\partial\alpha\partial\beta} + \beta'\frac{\partial^2 f}{\partial\beta^2} + \frac{\partial f_1}{\partial\beta}\right](z' - \beta x')}{(z' - \beta x')\frac{\partial f}{\partial\beta}},$$

which is independent of the point $x', y', z', 0$.

Therefore we have the theorem, that if tangent planes be drawn through any line in a given plane passing through a given point in the plane, the centre of mean position of the points of contact with reference to the plane, will be independent of the line. If the plane be the plane at infinity this becomes the well-known theorem, that the mean centre of the points of contact of parallel tangent planes to a surface is independent of the direction of the planes.

By reversing the process of elimination we get the co-ordinates of the mean centre of position of any line passing through the intersection of the x, z, and w planes, and lying in the w plane in a different form, by the solution of

$$x^n f + x^{n-1} w f_1 + \ldots = 0,$$

$$x^{n-1}\frac{\partial f}{\partial u} + x^{n-2} w \frac{\partial f_1}{\partial u} + \ldots = 0,$$

and
$$x^{n-1}\frac{\partial f}{\partial v} + x^{n-2} w \frac{\partial f_1}{\partial v} + \ldots = 0.$$

From these we obtain

$$-\Sigma\frac{x}{w} = \Sigma\frac{\alpha'\frac{\partial f}{\partial\alpha} + \beta'\frac{\partial f}{\partial\beta} + f_1}{f},$$

where α, β, α', and β' are determined by the equations

$$\frac{\partial f}{\partial\alpha} = 0, \qquad \frac{\partial f}{\partial\beta} = 0;$$

and
$$\alpha'\frac{\partial^2 f}{\partial\alpha^2} + \beta'\frac{\partial^2 f}{\partial\alpha\partial\beta} + \frac{\partial f_1}{\partial\alpha} = 0,$$

$$\alpha'\frac{\partial^2 f}{\partial\alpha\partial\beta} + \beta'\frac{\partial^2 f}{\partial\beta^2} + \frac{\partial f_1}{\partial\beta} = 0;$$

$$\therefore\quad -\frac{X}{W} = \frac{1}{n(n-1)^2}\,\Sigma\frac{f_1}{f}, \quad \text{since} \quad \frac{\partial f}{\partial\alpha} = \frac{\partial f}{\partial\beta} = 0.$$

We will now find the mean centre of position with respect to the plane of the poles of the plane with respect to the surface. The poles will be given by the solution of

$$x^{n-1}\frac{\partial f}{\partial u} + x^{n-2}w\frac{\partial f_1}{\partial u} + \ldots = 0\,,$$

$$x^{n-1}\frac{\partial f}{\partial v} + x^{n-2}w\frac{\partial f_1}{\partial v} + \ldots = 0\,,$$

and

$$x^{n-1}F + x^{n-2}wF_1 + \ldots = 0\,;$$

where

$$F = nf - u\frac{\partial f}{\partial u} - v\frac{\partial f}{\partial v}\,,$$

and

$$F_1 = (n-1)f_1 - u\frac{\partial f_1}{\partial u} - v\frac{\partial f_1}{\partial v}\,.$$

We have, then,

$$-\Sigma\frac{x}{w} = \Sigma\frac{\alpha'\dfrac{\partial F}{\partial \alpha} + \beta'\dfrac{\partial F}{\partial \beta} + F_1}{F}\,,$$

where α, α', β, and β' are determined by exactly the same equations as before. Substituting the values of F and F_1, we find

$$-\Sigma\frac{x}{w} = \frac{n-1}{n}\Sigma\frac{f_1}{f}\,;$$

$$\therefore\ -\frac{X}{W} = \frac{\dfrac{n-1}{n}\Sigma\dfrac{f_1}{f}}{(n-1)^3} = \frac{1}{n\,(n-1)^2}\Sigma\frac{f_1}{f}\,.$$

Therefore we have the theorem that the centre of mean position with respect to a plane of the points of contact of tangent planes to a surface through any line in the plane is independent of the line, and coincides with the mean centre of position with respect to the plane of the poles of the plane with respect to the surface. An obvious consequence is that the mean centre of the points of contact of parallel tangent planes coincides with the mean centre of the poles of the plane at infinity. As in the plane, this point is readily seen to be the tangential pole of the plane.

WOMAN'S COLLEGE, *March*, 1891.

ON THE DEPRESSION OF AN ALGEBRAIC EQUATION WHEN A PAIR OF ITS ROOTS ARE CONNECTED BY A GIVEN LINEAR RELATION.

By Mr. H. A. Sayre, Montgomery, Ala.

The usual method of depressing $f(x) = 0$, when a given linear relation exists between a and a', two of its roots, is to substitute for x the value of a' (expressed in terms of a); the result, considered as an equation for a, has a common root with $f(a) = 0$, and a can then be found by the greatest common measure method.

Sometimes, however, the following method is to be preferred: Let a and a' be roots of the quadratic $X = x^2 - bx + c = 0$; divide $f(x)$ by X, leaving a remainder of the form $\varphi x + \theta$; to make this vanish identically, equate φ and θ separately to zero, express them as functions of a by the relation $b = a + a'$, $c = aa'$, and the given relation $la + ma' = p$; apply the G. C. M. method to find a the common root of $\varphi = 0$, $\theta = 0$.

In the former method both equations are of the same degree (n suppose), while in the latter φ is of the $(n-1)$st degree in a and θ of the nth.

This will appear from the law of formation of the successive coefficients in the quotient and remainder arising from the division of $f(x)$ by $x^2 - bx + c$; it is evident from the process of division that each coefficient is equal to b times the preceding, minus c times the next preceding, plus the corresponding coefficient in the dividend, until we come to θ, in which case we omit the multiplier b. Now b, when expressed in terms of a, is of the first degree, and c is of the second; hence the second, third, fourth, etc. terms of the quotient will involve a in the first, second, third, etc. degrees; the last, or $(n-1)$st term, will involve a in the $(n-2)$nd degree, φ will be of the $(n-1)$st degree in a, and θ of the nth. The induction in this case is so evident that it need not be formally given.

If the G. C. M. of φ and θ be of the second degree, it will give two values of a, and there are two corresponding values of a', indicating that two pairs of the roots of $f(x) = 0$ are connected by the given relation; similarly if $\varphi = 0$, and $\theta = 0$, have s common roots, there are s pairs of the roots of $f(x) = 0$ connected by the same relation.

Suppose, for example, that we have the equation

$$x^4 - 7x^3 + 8x^2 + 7x + 15 = 0, \tag{1}$$

and that it is known that two of its roots a and a' are connected by the relation $2a' - 3a = 1$.

To solve by the usual method, we substitute $\frac{1}{2}(3a + 1)$ for x in (1); thus

$$27a^4 - 90a^3 - 12a^2 + 82a + 105 = 0; \tag{2}$$

and then find the G. C. M. of (1) and (2).

By the second method

$$\varphi = 65a^3 - 223a^2 + 59a + 75 = 0,$$
$$\theta = 57a^4 - 167a^3 - 5a^2 + 19a - 120 = 0;$$

and then apply the G. C. M. method as usual.

It may be noticed that the second method theoretically lessens the labor of obtaining the G. C. M., since φ is one degree lower than $f(a)$; practically, however, this is not always the case.

There is a case in which the usual method does not assist us in solving a proposed equation, namely, when we have an equation $f(x) = 0$, and it is known that the roots of this equation occur in pairs, and that *each* pair of roots a and a' satisfies the relation $a + a' = r$ (Todhunter, Theory of Equations, Art. 128).

The second method presents no such difficulty.

Suppose, for example, that we have the equation

$$f(x) = x^4 - 12x^3 + 13x^2 + 138x + 112 = 0,$$

where it is known that each pair of roots a and a' satisfies the relation $a + a' = 6$.

By the usual method we should proceed to investigate the common roots of the equations $f(x) = 0$ and $f(6 - x) = 0$. But these equations will be found to coincide completely, and do not afford a solution. The second method furnishes it easily. Write X in the form $x^2 - 6x + c$. $\varphi = 0$ vanishes, and $\theta = 0 = c^2 + 23c + 112$; whence by solving a quadratic equation we obtain c, and the roots are easily found. It may be remarked that when $f(x) = 0$ has equal roots, φ, expressed as a function of a, is the first derivative of $f(a)$.

EDDY'S SOLUTION OF A PROBLEM IN GRAPHICAL STATICS.*

In these two papers, Prof. Eddy, whose important contributions to the subject of Graphical Statics are familiar to American engineers and mathematicians, has developed at considerable length a branch of the subject which, though not entirely new, has been comparatively unexplored. The importance of these papers is such that they merit more than a passing notice.

As is well known, the use of actual concentrated loads in proportioning the members of a bridge is frequently specified by bridge engineers, and although this leads to an apparent accuracy which is for the most part illusory (owing to various necessary inaccuracies in our theories, the effect of shock and impact, and the fact that the heaviest actual loads vary from time to time), yet it is evident that such actual loads, whether used directly or indirectly, must at least form the *basis* of all such computations.

The principle upon which Prof. Eddy's method is based is the construction of a stepped line or *load line,* in which the horizontal distances between steps represent the distances apart of a system of loads, and the heights of the steps the loads themselves. From this load line, two irregular lines (reaction lines) may be drawn, one above and one below the load line. These reaction lines are parallel and at a constant horizontal distance apart, and they are so constructed that their vertical distances from the load line, at the ends of any span , represent the reactions at the ends of such a span in the assumed position. Reaction lines must be drawn for as many different spans as it is desired to treat.

By means of the load line and the reaction lines, almost all questions relating to shears and moments may be solved, such as the following: the reactions for any position of the given loads; the centre of gravity of the loads; the loads on the joints of a truss; the loading producing maximum shear at any point of a girder or in any panel of a truss, and the value of such shear; the segments of a span in which the maximum shear at any section is dominated by each successive wheel, or, in other words, occurs with that wheel at the section; the loading giving maximum moment at any

*Prof. Henry T. Eddy, Ph. D. A new Graphical Solution of the Problem, What position a Train of Concentrated Loads must have in order to cause the greatest stress in any given part of a Bridge Truss or Girder. (Transactions of the American Society of Civil Engineers, 1890.)

The same. Auflagerdrucklinien und deren Eigenschaften. (Zeitschrift für Bauwesen, 1890, pp. 397-415.)

point, together with the maximum moment itself, in a girder or truss. In fact, all the usual problems regarding moments and shears may be solved with ease and expedition, some from the load line alone, and others with the aid of the reaction lines, which are supposed drawn on a large diagram for many different spans. Some of the results reached are merely graphical representations of principles which were long ago known and used, while others are new.

The method used in this paper is not a purely graphical one, but rather an analytical one *interpreted graphically;* that is to say, analytical expressions are first deduced for the functions or conditions required and then, by algebraic transformations which are often somewhat complicated, but ingenious in the extreme, these expressions are brought into forms in which their terms are easily shown to represent certain distances, ratios, or angles on the diagram, and the analytical result is thus graphically interpreted.

In Part II of the paper, the author extends the method to finding shears, moments, and stresses in cantilevers and in double intersection trusses.

As a contribution to mechanics, these papers are very interesting, and will repay careful study. For the designing engineer, too, they offer many suggestions which will prove of value. Against the extended practical use of many of the theorems, however, one strong argument suggests itself. It is of the first importance for the computer, in working out his results, to use a method which is *simple*, and which allows him to see, at any stage, just where he is and what he is doing; otherwise he is liable to mistake, and an error may remain undiscovered. A method, therefore, which involves deducing an analytical expression, and then transforming it by an extended and complex process, which conveys no concrete idea but is merely a mathematical device, leads to a result the concrete relation of which to the problem cannot be perceived; and from this point of view many designers will much prefer to use analytical or other graphical methods (of which there are many) which afford a clearer view of the whole problem, and enable the computer to see the bearing of each step in the process. This criticism, however, does not apply to all the theorems in the paper, and the practical designer who takes the time to carefully study the subject will be well repaid.

GEO. F. SWAIN.

TISSERAND'S MÉCANIQUE CÉLESTE.* †

If one were asked to name the two most important works in the progress of mathematics and physics, the answer would undoubtedly be, the Principia of Newton and the Mécanique Céleste of Laplace. In their historical and philosophical aspects these works easily outrank all others, and furnish thus the standard by which all others must be measured. The distinguishing feature of the Principia is its clear and exhaustive enunciation of fundamental principles. The Mécanique Céleste, on the other hand, is conspicuous for the development of principles and for the profound generality of its methods. The Principia gives the plans and specifications of the foundation; the Mécanique Céleste affords the key to the vast and complex superstructure. It would be a mistake, of course, to suppose that Newton had no forerunners or that he had no worthy followers before Laplace. The continuity in the evolution of mechanical science is distinctly traceable, from the time of Galileo at any rate, down to the present day. But the Principia and the Mécanique Céleste present at once, more completely than any other works, the results of the great discoveries of their authors, and a perfect index to the state of contemporaneous theory. In addition, they present two distinct methods of investigation and exposition, and in this respect alone they merit the most attentive consideration.

It is natural, therefore, when a new treatise on celestial mechanics appears, to recur to the works of Newton and Laplace, to pass in review the peculiar features which render these works so specially prominent, and to align ourselves for the moment along the chain of history which connects these events with each other and with subsequent developments.

It was the happy lot of Newton to attain three brilliant achievements. First and greatest of these was the well-nigh perfect statement of the laws of dynamics; the second was the discovery of the law of gravitation; and the third was the invention of a calculus required to develop the consequences of the other two. It should be said, however, that the laws of motion were not unknown to the predecessors and contemporaries of Newton. Galileo, in fact, discovered the first two, and the third in one form or another was known to

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* Read before the Philosophical Society of Washington, April 25, 1891.

† Traité de Mécanique Céleste, par F. Tisserand. Paris, Gauthier-Villars et Fils. Tome I, Perturbations des planètes d'après la méthode de la variation des constantes arbitraires, 1889. Tome II, Théorie de la figure des corps célestes et de leur movement de rotation, 1891.

Hooke, Huyghens, and others; but it seems to have been the peculiar work of Newton to state these laws so clearly and fully, that the lapse of two centuries has suggested little, if any, improvement.

The law of gravitation, though commonly considered the greatest of Newton's achievements, is, in reality, less worthy of distinction than his foundation for dynamics. Its chief merit lay in the clear perception of the application of the law to the smallest particles of matter, for the mere notion of gravitation between finite masses was familiar to his contemporaries; in fact, according to Newton's own statement, the law of inverse squares as applicable to such masses was within the reach of any mathematician some years before the publication of the Principia.

The invention of the calculus, or the method of fluctions as Newton called it, was indeed a great achievement, much greater, probably, than the discovery of the law of gravitation. Unfortunately for science, however, and especially for British science, this new method of analysis figures as a silent partner in the Principia. The mathematical fashion or prejudice of Newton's day was strongly in favor of geometrical reasoning; and although the results of the Principia, as we now know, were derived by means of his calculus, he felt constrained to translate them into geometrical language. It was desirable, he thought, that the system of the heavens should be founded on good geometry. Subsequent history shows that this course was an ill-judged one. The geometrical method of the Principia renders it cumbersome, prolix, and on the whole rather repulsive to the modern reader; and the only justification which appears at all adequate for the exclusive adoption of this method, lies in the fact that his fellow countrymen would not have readily appreciated the more elegant and vastly more comprehensive analytical method. The result was very unfavorable to the growth of mathematical science in his own country. The seed he sowed took root on the continent, and has ever since grown best in French and German soil. According to Professor Glaisher, "the geometrical form of the Principia exercised a disastrous influence over mathematical studies at Cambridge University for nearly a century and a half, by giving rise to a mistaken idea of the relative power of analytical and geometrical processes."*

Readers of English mathematical text-books and treatises can hardly fail to notice the bias they show for geometrical methods, and especially for the formal, Euclidean mode of presentation, in which the procession of ideas con-

* From an address in commemoration of the bi-centenary of the publication of Newton's Principia, delivered at Cambridge University April 19, 1888. The only reference at hand to this important address is the Cambridge Chronicle and University Journal, Isle of Ely Herald, and Huntingdonshire Gazette, April 20, 1888.

sists too frequently of formidable groups of painfully accurate and technical propositions, corollaries, and scholiums. This formalism leads to a strained and unattractive literary style, which frequently degenerates into intolerable complexity or obscurity. Another and equally serious result of the apotheosis of pure geometry, is the tendency to magnify the importance of ideal problems and the work of problem solving. The exclusive pursuit of such aimless puzzles constitutes the platitude of mathematical research, though it often happens that the devotees to this species of work are mistaken for mathematicians and natural philosophers.

More than a hundred years elapsed between the publication of the Principia and the appearance of the first volumes of the Mécanique Céleste of Laplace. So slow, in fact, were mathematicians to appreciate the importance of Newton's discoveries that no improvements on his lunar and planetary theories were made before the middle of the 18th century. Continental mathematicians it would seem were obliged to develop for themselves the analytical method which Newton had used, but unwittingly discredited, before they were prepared to comprehend the scope of Newton's results. How thoroughly and completely they accomplished this work is attested by the contributions of Clairaut, Euler, D'Alembert, Legendre, Laplace, and Lagrange. The culmination of this preparatory work is exhibited more strikingly, perhaps, in the Mécanique Analytique of Lagrange, published just one hundred and one years after the Principia, than in any other single treatise. In the preface to this remarkable work Lagrange indicates in a single paragraph to what extent continental mathematicians had departed from the geometrical methods of investigation and exposition. "One will find," he says, "no diagrams in this work. The methods I expound require neither geometrical constructions nor geometrical reasoning, but only algebraical operations, subjected to a regular and uniform procedure. Those who love analysis will be pleased to see mechanics become a branch of it, and will wish me well in having thus extended its domain."

In addition to this emancipation from the narrowness of geometrical methods, continental mathematicians divested themselves from the formalism of Euclidean statement. They cultivated a literary style less distressingly precise and obtrusively technical, but immensely more luminous and attractive. The flow of ideas became with them easy and natural as well as logical; and it became less and less fashionable to inform the reader which of the ideas might be labled Proposition, Theorem, etc., or to inform him by means of the initials Q. E. D. when and where a chain of reasoning ended.

But this departure from the forms and methods of the Principia, shows

little if any detraction from the just merits of Newton's fame. Nearly every distinguished mathematician of the last generation has repeatedly acknowledged his debt and rendered his homage. They were neither blind nor servile, however, in their hero worship. They were too profoundly interested in the great problems of nature to be distracted by small issues. Animated by the zeal which such real problems impart to investigators, they sought to make their expositions comprehensive, clear, elegant, and attractive. Most of the great memoirs of this period were published in French, the language par excellence for mathematical exposition, and many of them present charms of style equal to if not surpassing those of the masters in lighter literature.

Such then was the attitude of mathematicians toward mechanical science at the end of the 18th century, when the first volumes of the Mécanique Céleste appeared. Laplace himself had done much to attain and secure this attitude, and was at this time probably the ablest investigator and expositor in the domains of mathematics and natural philosophy. The field on which he entered was a large one, and although it had been explored in many of its parts the work done was neither complete nor satisfactory. Observational astronomy and geodesy were pressing for more perfect theories and paving the way to their attainment with accumulating data. Above all, there was clearly recognized the need of a unification of the grand results which flow from the law of gravitation, and a complete exposition in one treatise of the chain of reasoning which connects the simplest concepts of matter and motion with the most complex phenomena of the material universe. This was the task which Laplace set for himself. Admirably equipped for the enterprise, and profoundly impressed with its magnitude and difficulties, he worked unceasingly for more than a quarter of a century in its accomplishment. The five volumes of the Mécanique Céleste, together with his Système du Monde, constitute the greatest systematic treatise ever published.

The treatise of Tisserand, which is the immediate object of this review, is a work in three quarto volumes, the first two of which have appeared. The first volume is devoted chiefly to the general theory of perturbations. The second volume treats of the figures of the planets and their movements of rotation. The third volume will be devoted to the theory of the moon, to an abridged theory of the satellites of Jupiter, to Hansen's method of computing perturbations, and to recent additions in celestial mechanics.

Without aiming to be thoroughly comprehensive the object of the author is elementary exposition. To those who would penetrate beyond the limits of his work, "into the more minute details of an arduous science," he tells us it goes without saying that the great treatise of Laplace will be found indispen-

sable. The author has, nevertheless, given us a very full presentation of many subjects and has devoted special attention to recent advances, so that the first two volumes bring the science substantially down to date.

We shall consider the details of the second volume only, glancing briefly at its several chapters seriatim.

The first three chapters are devoted to the general theory of attractions in conformity with the Newtonian law of inverse squares. It is worthy of remark that the potential function, which is the principal subject of exposition in these chapters, did not appear in analysis until ninety years after the publication of the Principia. Its discovery though generally credited to Laplace is due, so far as priority goes, at any rate, to Lagrange, who used it in the first instance, as others did for a long time, as a purely analytical device. The remarkable properties of this function are clearly and attractively derived by Tisserand. His account of the various theorems of Laplace, Poisson, Green, Gauss, and Chasles, cannot fail to impress the student with an appreciative sense of the profound knowledge attained by these masters, and also of the profounder ignorance which abides with us concerning the properties of matter. In reading these chapters one cannot avoid raising the query whether any additional generalizations are to be expected in this field. Those who look for such advances may find any tendency to overconfidence checked by the following brief historical summary of the principal events in the progress of the theory:

Newton, Law of Gravitation announced,	1687;
Lagrange, Potential Function introduced,	1777;
Laplace, Theorem or Equation introduced,	1782;
Poisson, extension of Laplace's Equation,	1813;
Green, Theorem and use of name Potential,	1828;
Gauss, General Theorems,	1840.

Chapter IV is devoted to the attraction of homogeneous ellipsoids, with special reference to the contributions of Lagrange, Ivory, and Gauss. Chapter V considers the attractions of homogeneous ellipsoids of revolution, together with attendant mathematical developments, and the attractions of some simple solids. Chapters VI and VII enter upon the more difficult enquiry of the equilibrium of homogeneous rotating fluid masses subject to gravitation. In these, along with applications to the sun and planets, considerable space is devoted to the Jacobian ellipsoid, which is remarkable as being a possible form of stable equilibrium, but of which no representative appears to have been discovered in nature.

At the close of the VIIth Chapter we have a most striking and suggestive theorem recently* deduced by M. Poincaré. A general criterion for the equilibrium of rotating fluid masses has long been needed, and Poincaré sought to supply this need by his theorem. It cannot be regarded in its present form as satisfactory, because it is not sufficiently precise; but it is astonishingly simple and may be susceptible of easy improvement. The result is derived by an application of Green's theorem, and as presented by Tisserand, is adapted to homogeneous masses only. But this restriction is unnecessary and does not lead to any material simplification. In its more general form the theorem may be stated thus: Let ω be the constant angular velocity and V the volume of any fluid mass M. Let u be the potential of the forces acting on a unit of this mass at any point of its surface, ∂n an element of the normal, and $d\sigma$ an element of the surface at the same point. Then, if we make one of the functions in Green's theorem equal to u and the other constant, and call f the unit of attraction, there results

$$2\omega^2 V - 4\pi f M = -\int \frac{\partial u}{\partial n}\, d\sigma .$$

Now this equation shows that in case

$$\omega^2 > 2\pi f \rho,$$

where $\rho = M/V$, or the mean density of the mass, the integral

$$\int \frac{\partial u}{\partial n}\, d\sigma$$

extended over the entire surface of M must be negative. In other words, the normal force $\partial u/\partial n$ must be directed outwards over some portions of that surface. Hence, the theorem asserts that equilibrium is impossible when the above inequality exists.

Obviously, the specification of a perfectly definite criterion is that $\partial u/\partial n$ shall be negative at *no* point of the surface. The attainment of this condition requires the removal of the integral sign in the second member of the above equation, or differentiation of the first member with respect to σ. Whether this is practicable, however, in any but simple cases is not evident.

Chapters VIII to XII treat of the equilibrium of rotating fluid masses, and especially of annular masses, subject to the attraction of adjacent bodies. They give an extended exposition of the researches of Laplace, Maxwell, Madame Kowalewski, and Poincaré concerning the form and stability of the

* 1885, Bulletin astronomique, t. II, p. 117.

rings of Saturn. The interesting details of this subject need not be dwelt on here. We shall only quote the opinion of Tisserand relative to Maxwell's work on the rings of Saturn for the benefit of those who have found Maxwell's Electricity and Magnetism unsatisfactory reading. After explaining the general process and basis of Maxwell's investigation, Tisserand says: "The object of the memoir of Maxwell is very important, but the reasoning lacks rigor, precision, and especially clearness, and we find ourselves obliged, to our great regret, to reproduce his conclusions only."

Chapter XIII enters upon the real problem presented by the planets of a heterogeneous, rotating, spheroidal mass. It gives an admirable exposition of the theory of Clairaut founded on the recent researches of M. Hamy. Much space is devoted in Chapters XIV and XV to the investigations of Radau, Poincaré, Callandreau and others relative to the possible arrangement of density in the earth. The most important conclusion from these investigations is that if the density of the earth increases continuously from surface to centre, in whatever way, and if the surface shape of the earth is due to its original fluidity, then the surface flattening cannot exceed $\frac{1}{297.3}$.

Chapters XVI to XIX are devoted to the theory of the figures of the planets based on spherical harmonic analysis. This is the great theory of Laplace, and the chapters in question are virtually a reproduction of Laplace's work, together with an excellent exposition of the remarkable properties of his harmonic functions. The last of these chapters closes with an elegant demonstration, due to Poincaré, of the theorem of Stokes. This theorem is probably the most important addition to terrestrial mechanics made since the time of Laplace and must be considered one of the most surprising results of analysis. It amounts to saying that the potential of a planet, like the earth, rotating about a fixed axis, at any external point is determined by the shape of its sea surface, irrespective of the arrangement of the constituents of its mass. The demonstration of Poincaré is accomplished by an application of the ever-fruitful theorem of Green.

Chapters XX and XXI give a brief account of the theories of geodesy, with special reference to the determination of the dimensions and figure of the earth by means of arc measures and by means of the pendulum. In connection with the latter subject a short account is given of Helmert's condensation theory for the treatment of pendulum observations. These chapters are confessedly incomplete, but a fuller exposition was hardly desirable in view of the fact that the whole subject has been treated recently in a very thorough manner by Helmert in his admirable work entitled Die Mathematischen und Physikalischen Theorieen der Höheren Geodäsie.

Chapters XXII to XXVII are devoted to the motions of rotation of the celestial bodies and to the problems of precession and nutation which arise from the interaction of those bodies. These chapters, like Chapter XXVIII, which treats of the libration of the moon, are of interest chiefly to the specialist in physical astronomy, though well worthy of the attention of students in allied fields of research. Chapter XXIX is of more general interest, since it considers the influence of geological action on the rotation of the earth, gives an account of the researches of Hopkins, Sir William Thomson, Darwin, and others in this field, and treats at some length the question of variations of terrestrial latitudes, which is now one of special importance to astronomers and geodesists.

The last chapter treats of the rotations of bodies of variable form, of the effects of tidal friction, and of various allied questions which are not yet fully settled.

On the whole, for the ground covered by the two volumes already in print, the treatise of Tisserand appears to give the best exposition of celestial mechanics at present available. The arrangement and development of the subject are clear and orderly. The notation, which is a feature of the highest importance in such a work, conforms in general with the best modern usage. The typography has the characteristic excellence of the famous house of Gauthier-Villars et Fils, and the work has evidently been proof read with the greatest care. The specialist must, of course, consult many other treatises and memoirs, and particularly the great work of Laplace. But if to such the Mécanique Céleste is indispensable, the Traité of Tisserand can fall little short of a necessity for the reason that it records the growth of the science since the epoch of Laplace, and places the reader in line with the march of future improvements.

R. S. Woodward.

SOLUTIONS OF EXERCISES.

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A BODY is projected at an angle of 30° with the horizon, with a given velocity. Determine the constant resistance it must suffer in the direction contrary to its motion in order that it may come to rest when it returns to the horizontal plane whence it started. Also determine the horizontal range, time of flight, and length of trajectory. [*Jas. M. Ingalls.*]

SOLUTION.

Let V be the initial velocity and φ the inclination of the tangent to the horizon, at a point of the path where the velocity of the projectile is v. Let r be the constant resistance, and for convenience let the mass of the projectile be unity.

Resolving along the tangent, we have

$$\frac{dv}{dt} = -r - g \sin \varphi \,;$$

along the normal

$$\frac{v^2}{\rho} = g \cos \varphi \,, \quad \text{whence} \quad dt = -\frac{v d\varphi}{g \cos \varphi}\,.$$

These give

$$\frac{dv}{v} = \frac{r}{g} \sec \varphi \, d\varphi + \tan \varphi \, d\varphi\,.$$

Integrating this between v and V, we have, if α be the angle of projection,

$$v = V \frac{\cos \alpha}{\cos \varphi} \frac{\tan^{\frac{r}{g}} (\frac{1}{4}\pi + \frac{1}{2}\varphi)}{\tan^{\frac{r}{g}} (\frac{1}{4}\pi + \frac{1}{2}\alpha)}\,.$$

Since at the finish v is zero, this ratio vanishes for $\varphi = -\frac{1}{2}\pi$, which is the inclination of the terminal tangent to the horizon. (In its present shape the ratio is $0/0$ for $\varphi = -\frac{1}{2}\pi$, but the ratio of the derivatives of the numerator and denominator vanishes for $\varphi = -\frac{1}{2}\pi$.)

Again, the change of energy is the work done; hence

$$\begin{aligned} v dv &= (-r - g \sin \varphi)\, ds \\ &= -r ds - g dy. \end{aligned}$$

Integrating, from start to finish, we have

$$V^2 = 2rS,$$

S being the length of the trajectory.

The rise, range, time of flight, and length of trajectory are to be determined from the formulæ

$$Y = \frac{1}{g}\int_0^\alpha v^2 \tan\varphi\, d\varphi\,, \qquad (1) \qquad\qquad T = \frac{1}{g}\int_{-\frac{1}{2}\pi}^\alpha v \sec\varphi\, d\varphi\,, \qquad (3)$$

$$X = \frac{1}{g}\int_{-\frac{1}{2}\pi}^\alpha v^2\, d\varphi\,, \qquad (2) \qquad\qquad S = \frac{1}{g}\int_{-\frac{1}{2}\pi}^\alpha v^2 \sec\varphi\, d\varphi\,; \qquad (4)$$

in which the values of v as determined above in terms of φ must be substituted.

There are two ways offered for determining r: we may put $-\frac{1}{2}\pi$ for the lower limit in the integral (1) instead of 0 and equate the result to zero and solve for r; otherwise equate the values of S in (4) to that in $S = V^2/2r$ and solve the result for r. Either method gives $r = \frac{5}{4}g$.

Thus by the former, put $2r/g = n$, $\frac{1}{4}\pi + \frac{1}{2}\alpha = \beta$, and $\frac{1}{4}\pi + \frac{1}{2}\varphi = \theta$ for brevity; then

$$\int_{-\frac{1}{2}\pi}^\alpha \tan^{\frac{2r}{g}}(\tfrac{1}{4}\pi + \tfrac{1}{2}\varphi)\sec^2\varphi \tan\varphi\, d\varphi = 0$$

becomes

$$\int_0^{\tan\beta}(\tan^{n-3}\theta - \tan^{n+1}\theta)\, d(\tan\theta) = 0\,,$$

or

$$\frac{\tan^{n-2}\beta}{n-2} = \frac{\tan^{n+2}\beta}{n+2}\,.$$

Whence

$$n = 2\,\frac{\tan^4\beta + 1}{\tan^4\beta - 1}\,.$$

This gives $r = \frac{5}{4}g$, since $\alpha = 30°$ and $\beta = 60°$.

$$\therefore\ S = \frac{2}{5}\,\frac{V^2}{g}\,.$$

From the same integral we have for the rise, putting

$$K = \frac{V^2}{g}\,\frac{\cos^2 30°}{\tan^{\frac{5}{2}} 60°} = \frac{1}{4\sqrt[4]{3}}\,\frac{V^2}{g}$$

for shortness,

$$Y = \frac{1}{g}\int_0^\alpha v^2\tan\varphi\, d\varphi = -\frac{1}{4}K\int_1^{\sqrt{3}}(\tan^{-\frac{1}{2}}\theta - \tan^{\frac{7}{2}}\theta)\, d(\tan\theta)$$

$$= -\frac{1}{4}K\left\{2\tan^{\frac{1}{2}}\theta - \frac{2}{9}\tan^{\frac{9}{2}}\theta\right\}_1^{\sqrt{3}} = \frac{1}{9\sqrt[4]{3}}\,\frac{V^2}{g}\,.$$

In like manner

$$X = \frac{1}{g}\int_{-\frac{1}{2}\pi}^\alpha v^2 d\varphi = \frac{1}{2}K\int_0^{\sqrt{3}}(\tan^{\frac{1}{2}}\theta + \tan^{\frac{5}{2}}\theta)\, d(\tan\theta)$$

$$= K\left\{\frac{1}{3}\tan^{\frac{3}{2}}\theta + \frac{1}{7}\tan^{\frac{7}{2}}\theta\right\}_0^{\sqrt{3}} = \frac{4\sqrt[4]{3}}{21}\,\frac{V^2}{g}\,.$$

Finally

$$T = \frac{1}{g}\int_{-\frac{1}{2}\pi}^{a} v \sec\varphi \, d\varphi = \frac{V}{g}\frac{\cos 30^\circ}{\tan^{\frac{5}{2}} 60^\circ}\int_{-\frac{1}{2}\pi}^{a} \tan^{\frac{3}{2}}(45^\circ + \tfrac{1}{2}\varphi)\sec^2\varphi \, d\varphi$$

$$= \frac{1}{3^{\frac{5}{4}}}\frac{V}{4g}\int^{\sqrt{3}}(\tan^{\frac{3}{2}}\theta + \tan^{\frac{1}{2}}\theta)\, d(\tan\theta) = \frac{44\sqrt{3}}{153}\frac{V}{g}.$$

[*W. H. Echols.*]

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Two sides of a triangle are a and b; find the average length of the third side. [*Artemas Martin.*]

SOLUTION.

I. Supposing the third side to vary uniformly, let $x =$ the third side; then it is obvious that $a + b$ is the *greatest* value x can have, and $a - b$ is the *least* value x can have; and $\frac{1}{2}[(a + b) + (a - b)] = a$ is the average length of the third side. *Otherwise.* Let $x =$ third side, and $\varDelta =$ its average length; then

$$\varDelta = \int_{a-b}^{a+b} x\,dx \div \int_{a-b}^{a+b} dx, = a, \text{ as before.}$$

II. Supposing the angle included by the two given sides to vary uniformly, let φ be that angle; then $x = \sqrt{(a^2 + b^2 - 2ab\cos\varphi)}$, and

$$\varDelta = \int_0^{\pi}\sqrt{(a^2 + b^2 - 2ab\cos\varphi)}\, d\varphi \div \int_0^{\pi} d\varphi$$

$$= \frac{1}{\pi}\int_0^{\pi}\sqrt{(a^2 + b^2 - 2ab\cos\varphi)}\, d\varphi.$$

Let $\varphi = \pi - 2\theta$, then $\cos\varphi = -\cos 2\theta = 2\sin^2\theta - 1$, $d\varphi = -2d\theta$, and by substitution

$$\varDelta = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\sqrt{(a^2 + b^2 + 2ab - 2ab\sin^2\theta)}\, d\theta$$

$$= \frac{2(a + b)}{\pi}\int_0^{\frac{1}{2}\pi}\sqrt{\left[1 - \frac{2ab}{(a + b)^2}\sin^2\theta\right]}\, d\theta$$

$$= \frac{2(a + b)}{\pi}\mathrm{E}\left[\frac{\sqrt{2ab}}{a + b}, \tfrac{1}{2}\pi\right].$$

[*Artemas Martin.*]

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The points P, Q, R, S, T, U are the vertices of a regular hexagon inscribed in a circle with centre O. Join PR, cutting OQ in A; AS, cutting OR in B; BT, cutting OS in C; etc. Prove that OA, OB, OC, OD, etc. are proportional to the reciprocals of the natural numbers 1, 2, 3, 4, etc. [*Yale.*]

SOLUTION.

If $OP = R$, $OA = \frac{1}{2}R$. The triangles, AOB and RBS are similar; $\therefore OB = \frac{1}{2}BR = \frac{1}{3}R$. The triangles OBC and TCS are similar and $OB = \frac{1}{3}R = \frac{1}{3}ST$; $\therefore OC = \frac{1}{3}CS = \frac{1}{4}R$. And so on.

$\therefore OP : OA : OB : OC : \ldots :: 1 : \frac{1}{2} : \frac{1}{3} : \frac{1}{4} : \ldots$ [*W. B. Richards.*]

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If a triangle be cut out of paper and doubled over so that the crease passes through the centre of the circumscribing circle and the angle A, the area of the triangular double-part will be

$$\tfrac{1}{2}b^2 \sin^2 C \cos C \operatorname{cosec}(2C - B) \sec(C - B);$$

the angle C being greater than B. [*Yale.*]

SOLUTION.

Call O the centre of the circumscribed circle. OA cuts BC in D. Make $\angle ADF = \angle ADC$. DF cuts AB in F. Take DE on $DC = DF$. Then DAE is the triangle required.

We know that $\angle ADE = 90° - (C - B)$, and $\angle BAD = DAE = 90° - C$; also

$$\text{area } DAE = \tfrac{1}{2}\,\frac{AD^2 \sin DAE \sin ADE}{\sin(DAE + ADE)}.$$

But $AD = \dfrac{b \sin C}{\sin ADE}$;

$$\therefore \text{area } DAE = \tfrac{1}{2}\,\frac{b^2 \sin^2 C \sin DAE}{\sin ADE \sin(DAE + ADE)}$$

$$= \tfrac{1}{2}b^2 \sin^2 C \cos C \sec(C - B) \operatorname{cosec}(2C - B).$$

[*T. U. Taylor.*]

EXERCISE.

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A hollow sphere, external and internal radii R and r, and specific gravity s, is partly filled with water, and floats in a pond, the water in the sphere being on a level with the surface of the pond. Find the quantity of water in the hollow sphere. [*Artemas Martin.*]

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